

A General Framework for Codes Involving Redundancy Minimization

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Abstract—A framework with two scalar parameters is introduced for various problems of finding a prefix code minimizing a coding penalty function. The framework encompasses problems previously proposed by Huffman, Campbell, Nath, and Drmota and Szpankowski, shedding light on the relationships among these problems. In particular, Nath’s range of problems can be seen as bridging the minimum average redundancy problem of Huffman with the minimum maximum pointwise redundancy problem of Drmota and Szpankowski. Using this framework, two linear-time Huffman-like algorithms are devised for the minimum maximum pointwise redundancy problem, the only one in the framework not previously solved with a Huffman-like algorithm. Both algorithms provide solutions common to this problem and a subrange of Nath’s problems, the second algorithm being distinguished by its ability to find the minimum variance solution among all solutions common to the minimum maximum pointwise redundancy and Nath problems. Simple redundancy bounds are also presented.

Index Terms—Huffman algorithm, minimax redundancy, optimal prefix code, Rényi entropy, unification.

I. INTRODUCTION

A source emits symbols drawn from the alphabet $\mathcal{X} = \{1, 2, \dots, n\}$. Symbol i has probability p_i , thus defining probability mass function vector \mathbf{p} . We assume without loss of generality that $p_i > 0$ for every $i \in \mathcal{X}$, and that $p_i \leq p_j$ for every $i > j$ ($i, j \in \mathcal{X}$). The source symbols are coded into binary codewords. Each codeword c_i corresponding to symbol i has length l_i , thus defining length vector \mathbf{l} .

It is well known that Huffman coding [1] yields a prefix code minimizing $\sum_{i \in \mathcal{X}} p_i l_i$ given the natural coding constraints: the integer constraint, $l_i \in \mathbb{Z}_+$, and the Kraft (McMillan) inequality [2]:

$$\sum_{i \in \mathcal{X}} 2^{-l_i} \leq 1.$$

Hu, Kleitman, and Tamaki [3] and Parker [4] independently examined other cases in which Huffman-like algorithms were optimal; this work was later extended [5], [6]. Other modifications of the Huffman coding problem were considered in analytical papers [7]–[9], although none of these proposed a Huffman-like algorithmic solution. In each paper, relationships between the modified problem and the Huffman coding problem were

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explored. Parker proposed an algorithmically-motivated two-function parameterization defining various Huffman coding problems; these two parameter functions are a “weight combination” function and a “tree cost” function [4]. Three problems, first examined in [1], [7], [8], were considered as a part of this framework; here we show that a fourth [9] fits into it as well. In addition, we find a simpler redundancy-motivated unifying problem class that relates the four problems, one involving two scalar parameters rather than two functional parameters. This new framework reveals a united analytical structure, including simple redundancy bounds and novel algorithmic results which improve upon the algorithm of [9].

In Section II, background is given on the coding problem introduced in [7]. In Section III, the new framework, based on an extension of this problem, is introduced. The problem and the three other aforementioned problems are then put into the context of this framework. In Section IV, the framework is used to help find linear-time algorithms for the problem in [9]. Redundancy bounds are presented in Section V, with concluding thoughts following in Section VI.

II. BACKGROUND: EXPONENTIAL HUFFMAN CODING

One particular application of a modified coding problem was found by Humblet [10] for a problem involving minimization of buffer overflow in communications. In this application, the function minimized is $\sum_{i \in \mathcal{X}} p_i 2^{\beta l_i}$ for a given $\beta > 0$. This is easily generalized to negative β by specifying minimization of the β -exponential average

$$F_\beta(\mathbf{p}, \mathbf{l}) \triangleq \frac{1}{\beta} \log_2 \sum_{i \in \mathcal{X}} p_i 2^{\beta l_i}. \quad (1)$$

This problem was originally proposed by Campbell [7] and a linear-time algorithm found independently by Hu et al. in [3, p. 254], Parker in [4, p. 485], and Humblet in [11, p. 25] (later published as [10, p. 231]). This algorithm covers all of \mathbb{R} ; the case of $\beta = 0$ is considered by noting that $\beta \rightarrow 0$ yields the original Huffman coding problem.

Below is the procedure for the exponential extension of Huffman coding with parameter β . Note that it minimizes (1) over \mathbf{l} , even if the “probabilities” do not add to 1. We refer to such arbitrary positive inputs as *weights*, often denoted by $\mathbf{w} = \{w_i\}$ instead of $\mathbf{p} = \{p_i\}$:

Procedure for Exponential Huffman Coding

- 1) Each item $m_i \in \{m_1, m_2, \dots, m_n\}$ has weight $w_i \in \mathcal{W}_\mathcal{X}$, where $\mathcal{W}_\mathcal{X}$ is the set of all such weights. (Initially, $m_i = i$.) Assume each item m_i has codeword c_i , to be determined later.
- 2) Combine the items with the two smallest weights w_j and w_k into one item \tilde{m}_j with the combined weight $\tilde{w}_j = 2^\beta (w_j + w_k)$. This item has codeword \tilde{c}_j , to be determined later, while m_j is assigned code $c_j = \tilde{c}_j 0$

and m_k code $c_k = \tilde{c}_j 1$. Since these have been assigned in terms of \tilde{c}_j , replace w_j and w_k with \tilde{w}_j in \mathcal{W} to form $\mathcal{W}_{\tilde{\mathcal{X}}}$.

- 3) Repeat procedure, now with the remaining $n - 1$ codewords and corresponding weights in \mathcal{W} , until only one item is left. The weight of this item is $\sum_{i \in \mathcal{X}} w_i 2^{\beta l_i}$. All codewords are now defined by assigning the null string to this trivial item.

This algorithm can be modified to run in linear time (to input size) given sorted weights in the same manner as Huffman coding [12]. An example of exponential Huffman coding for $\beta = \log_2 1.1$ is shown in Figure 1. The resulting code is different from that which would be obtained via Huffman coding ($\beta = 0$).

The output of a Huffman-like algorithm might be a code (and thus the implicit code tree, e.g., [13]) or merely codeword lengths; we assume the latter from here on, because valid codewords can be inferred from the lengths. Thus we can view the problem such an algorithm solves as an integer optimization problem. This is useful because many different codes can correspond to the same set of codeword lengths and thus all be optimal for a given problem.

Considering the codeword lengths alone as the solution to a given problem, we find that some problems have a unique optimizing set of lengths, while others have more than one distinct optimal solution. Multiple different solutions manifest themselves in the algorithm as possible ties in the weight of (possibly combined) items in the combination step (step 2 above). Thus the algorithm, as with Huffman coding, is nondeterministic. Two deterministic variants are *bottom-merge Huffman coding* and *top-merge Huffman coding* [13]. Code trees yielded from the former method have been called, depending on the properties focused upon, *best Huffman trees* [14], *compact Huffman trees* [15], *minimal Huffman trees* [16], and *minimum variance Huffman trees* [17], the last of these because variance is minimized among (tied) optimal code trees (codeword lengths).

Given $b \in \mathbb{R}$ and \mathbf{p} , if we relax the integer constraint on \mathbf{l} , minimizing $F_b(\mathbf{p}, \mathbf{l})$ becomes a simple numerical optimization and provides a lower bound for the integer-valued problem. (We use b instead of β from here on to refer to the parameter for the real-valued problem.) Campbell [7] noted that the optimal value of $F_b(\mathbf{p}, \mathbf{l})$ for $b \in (-1, +\infty) \setminus \{0\}$ is the Rényi entropy of order $\alpha = (1 + b)^{-1}$:

$$\begin{aligned} H_\alpha(\mathbf{p}) &\triangleq \frac{1}{1-\alpha} \log_2 \sum_{i \in \mathcal{X}} p_i^\alpha \\ &= \frac{1+b}{b} \log_2 \sum_{i \in \mathcal{X}} p_i^{(1+b)^{-1}}. \end{aligned} \quad (2)$$

This should not be surprising given the relationship between Huffman coding and Shannon entropy, which corresponds to $b \rightarrow 0$, $H_1(\mathbf{p})$ [18].

Given $b \in (-1, +\infty)$ and \mathbf{p} , the optimal ideal real-valued

lengths achieving (2) are given by

$$l_i^\dagger = -\frac{1}{1+b} \log_2 p_i + \log_2 \sum_{j \in \mathcal{X}} p_j^{\frac{1}{1+b}}. \quad (3)$$

At the extremes of the $(-1, +\infty)$ range, solutions are defined as the limit of the solutions for $b \downarrow -1$ and $b \uparrow +\infty$, respectively. For $b < -1$, there is no real-valued solution, the problem being optimized by $l_1^\dagger = 0$ and $l_i^\dagger = +\infty$ for every $i > 1$.

III. MINIMIZATION OF d -AVERAGE b -REDUNDANCY

We call the difference between an integer l_i and the optimal real-valued solution l_i^\dagger the *pointwise b -redundancy*

$$r_b(i) \triangleq l_i - l_i^\dagger,$$

to emphasize its dependence on b . The arithmetic average of pointwise 0-redundancy was the problem considered by Huffman in his original paper, “A Method for the Construction of Minimum-Redundancy Codes.” Here we introduce a generalization of this problem encompassing several cases of interest.

Suppose we wish to minimize *d -average b -redundancy* or *DABR*,

$$R_{b,d}(\mathbf{p}, \mathbf{l}) \triangleq \frac{1}{d} \log_2 \sum_{i \in \mathcal{X}} p_i 2^{dr_b(i)}. \quad (4)$$

This amounts to finding $\mathbf{l}_{b,d}^*(\mathbf{p})$ such that

$$\begin{aligned} R_{b,d}(\mathbf{p}, \mathbf{l}_{b,d}^*(\mathbf{p})) &= \min_{\mathbf{l}} R_{b,d}(\mathbf{p}, \mathbf{l}) \\ &= \min_{\mathbf{l}} \frac{1}{d} \log_2 \sum_{i \in \mathcal{X}} p_i 2^{d(l_i - l_i^\dagger)} \\ &= \min_{\mathbf{l}} \frac{1}{d} \log_2 \sum_{i \in \mathcal{X}} \frac{p_i^{\frac{1+b+d}{1+b}}}{\sum_{j \in \mathcal{X}} p_j^{\frac{1+b+d}{1+b}}} 2^{dl_i} \end{aligned} \quad (5)$$

where \mathbf{l} is restricted to the integers and by the Kraft inequality (implicit from here on).

This reduces to an exponential Huffman coding problem. Then, given sorted $\{p_i\}$, (5) is solvable in linear time; note that the normalization of the terms is optional for the algorithm. For $d < -1$, the solution is always the unary code $\mathbf{l} = (1, 2, \dots, n-1, n-1)$. Considering the edges via limiting (as we did with real-valued solutions), the range of nontrivial cases for minimal DABR codes for a given probability mass function can thus be considered to be parameterized by $b \times d \in [-1, +\infty] \times [-1, +\infty]$, as in Figure 2.

As indicated in this figure, many interesting coding problems fit within this framework. These problems, which we discuss below, correspond to subsets of this two-dimensional extended quadrant $([-1, +\infty] \times [-1, +\infty])$. On the set of points for which b is $+\infty$, for example, the minimization reduces to exponential Huffman coding with parameter $\beta = d$. For $d = 0$ ($b \in (-1, +\infty)$) we have Huffman coding. A particular type

Codeword length	Codeword	Item	Weight
2	00	m_1	0.36
2	01	m_2	0.30
2	10	m_3	0.20
2	11	m_4	0.14

Fig. 1. Exponential Huffman coding for weights $\mathbf{w} = (0.36, 0.30, 0.20, 0.14)$ and $\beta = \log_2 1.1$. In the first step, the two smallest items (with weights $w_3 = 0.20$ and $w_4 = 0.14$) are combined into a compound item with weight $0.374 = 1.1 \cdot (0.20 + 0.14)$; thus c_3 and c_4 end in 0 and 1, respectively. At each additional step, the two smallest remaining items are combined in a similar fashion. In this manner a code to optimize (and as a by-product calculate) the value of $\sum_{i \in \mathcal{X}} w_i 2^{\beta l_i}$ is built from the bottom up. In this case, the minimized value is 1.21.

of Huffman coding occurs for $b = +\infty$, $d \downarrow 0$. In such a case, we note that

$$R_{b,d}(\mathbf{p}, \mathbf{l}) = \sum_{i \in \mathcal{X}} p_i l_i + \frac{d}{2} \sigma_{\mathbf{p}}^2(\mathbf{l}) + O(d^2) \quad \text{as } d \rightarrow 0 \quad (6)$$

where the second term on the right-hand side represents variance. This being the tie-breaking term, we have bottom-merge Huffman coding.

IV. MINIMIZATION OF MAXIMAL POINTWISE REDUNDANCY

As average pointwise (0-)redundancy has been well understood for some time, Drmota and Szpankowski decided to explore the previously overlooked minimization of maximal pointwise redundancy [9], [19].

We define d th exponential redundancy as DABR for $b = 0$. Note that the maximal redundancy problem is equivalent to minimizing d th exponential redundancy as $d \rightarrow +\infty$. Thus, considering $d \in [0, +\infty]$, d th exponential redundancy is a subproblem with a parameter that varies solution values between minimizing average redundancy (Huffman coding) and minimizing maximal redundancy; such a range of problems and solutions was sought in [19]. This was previously derived axiomatically without regard to such a range and without solution [8]. The version of the minimal DABR coding solution applying to the maximal redundancy subproblem was found shortly thereafter [4], although it was not generalized to $b \neq 0$ or to $d = +\infty$.

Drmota and Szpankowski presented a simple method for finding a code with minimum maximal redundancy [9], [19]. However, this solution is deficient in the following senses: First, time complexity is $O(n \log n)$. Second, the Kraft inequality is not necessarily satisfied with equality, meaning that the optimal code found in this manner is often, in some sense, wasteful. Third, the code does not necessarily optimize d th exponential redundancy for any $d < +\infty$. The method is also not generalized to maximal b -redundancy ($b \neq 0$).

In order to overcome the first two deficiencies, we propose a reduction to a previously-known algorithm with linear complexity previously discussed by Parker [4]. This problem was termed the *tree-height measure* problem, though it was not previously considered in the context of the maximal redundancy or DABR problems.

The tree-height measure problem minimizes the maximum value of $w_i + c \cdot l_i$ given $c > 0$ and weight vector \mathbf{w} . Instead of using $\tilde{w}_j = 2^{\beta}(w_j + w_k)$ on the merge step of Huffman coding, the Huffman-like tree-height measure algorithm uses $\tilde{w}_j = c + \max(w_j, w_k)$. In order to use the tree-height measure algorithm, assign weights according to

$$w_i(b) = \frac{1}{1+b} \log_2 \frac{p_i}{p_n},$$

which is always nonnegative, and let $c = 1$. Then this modified Huffman algorithm minimizes

$$\begin{aligned} \max_i (w_i(b) + c \cdot l_i) &= \max_i \left(l_i + \frac{1}{1+b} \log_2 \frac{p_i}{p_n} \right) \\ &= \max_i r_b(i) + \log_2 \sum_{j \in \mathcal{X}} p_j^{\frac{1}{1+b}} \\ &\quad - \log_2 p_n^{\frac{1}{1+b}}. \end{aligned}$$

Thus this linear-time algorithm returns a length vector minimizing maximum b -redundancy and satisfying the Kraft inequality with equality.

Because ties can occur in selecting weights to combine, the exponential Huffman algorithm might yield one of many possible optimal codes, including codes not optimal for the limit of d th exponential redundancy (as $d \rightarrow +\infty$). For example, consider $\mathbf{p} = (\frac{8}{19}, \frac{4}{19}, \frac{3}{19}, \frac{2}{19}, \frac{2}{19})$. For d th exponential redundancy, $\mathbf{l} = (1, 2, 3, 4, 4)$ and $\mathbf{l} = (1, 3, 3, 3, 3)$ are both optimal for $d \rightarrow +\infty$. These not only minimize maximal redundancy, but, among codes that optimize this, these codes also have the lowest probability of achieving this maximal redundancy, as this is related to the second term of the expansion of $R_{b,d}(\mathbf{p}, \mathbf{l})$

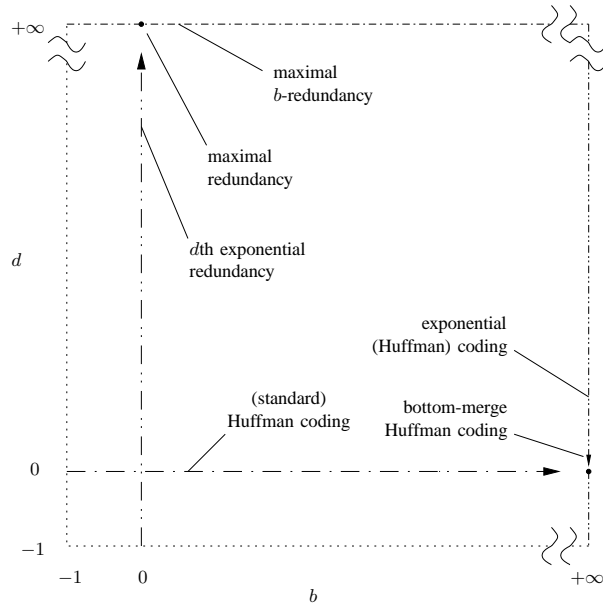


Fig. 2. The parameter space for minimal d -average b -redundancy (DABR) coding with the following noted subproblems: the Huffman problem (the above line at $d = 0$), Campbell's exponential coding problem (line at $b = +\infty$), the problem solved via Schwartz's bottom-merge Huffman coding method (limit point at $(+\infty, 0)$ when approached from above), Nath's d th exponential redundancy problem (line at $b = 0$), Drmota and Szpankowski's maximal redundancy (point at $(0, +\infty)$), and maximal b -redundancy (line at $d = +\infty$). Each point in the extended quadrant represents a different (parameterized) problem, as in Figure 5.

for $d \rightarrow +\infty$:

$$\begin{aligned} R_{b,d}(\mathbf{p}, \mathbf{l}) &= \frac{1}{d} \log_2 \sum_{i \in \mathcal{X}} p_i 2^{dr_b(i)} \\ &= \max_i r_b(i) \\ &\quad + \frac{1}{d} \log_2 P_X[r_b(X) = \max_j r_b(j)] \\ &\quad + O\left(\frac{1}{d^2}\right) \quad \text{as } d \rightarrow +\infty \end{aligned} \quad (7)$$

Each term in the expansion has a different asymptotic complexity. As with minimum variance (bottom-merge) Huffman coding (6), each additional term further restricts the set of feasible codes to those that minimize the current term given the optimization of previous terms. In the above example, all terms are minimized by both the aforementioned sets of lengths. In contrast, $\mathbf{l} = (2, 2, 2, 3, 3)$, although also minimizing maximal redundancy, results in a code where codewords have a higher probability of achieving maximal redundancy. This solution, which is in some sense inferior, can nevertheless be achieved by the tree-height measure algorithm, specifically the bottom-merge version.

It is possible to find a $D \in \mathbb{R}$ such that, for every $d \geq D$, d th exponential Huffman coding minimizes maximal redundancy. Let $\min_{i,j}^+ \gamma_{i,j}$ denote the minimum strictly positive value of $\gamma_{i,j}$, and let $\langle x \rangle$ denote the fractional part of x , i.e., $\langle x \rangle \triangleq x - \lfloor x \rfloor$. Assign $\delta = \min_{i,j}^+ (\langle l_i^\dagger - l_j^\dagger \rangle)$. It is possible to show

that a sufficiently large D is given by $D = \frac{1}{\delta} \log_2 \frac{2}{p_n} > 1$ [20, pp. 59-62]. However, finding D requires sorting, so an algorithm derived from this D would not be a linear-time algorithm.

Fortunately, it is possible to arrive at a linear-time algebraic Huffman algorithm, that is, one that keeps D as a variable. Algebraic Huffman algorithms were introduced by Knuth [5]. The one proposed here uses a Huffman algorithm which keeps track of both the first- and second-order terms; ties between these pairs of terms can occur only when all terms are tied, this due to the manner in which the Huffman procedure works. Before explaining why this is the case, we present the algorithm.

The aforementioned first- and second-order terms are

$$w'_i \triangleq \lim_{d \rightarrow +\infty} [w_i(b, d)]^{(d^{-1})}$$

and

$$w''_i \triangleq \lim_{d \rightarrow +\infty} [w_i(b, d)]^{-1} \cdot [w'_i]^d,$$

respectively, where leaf nodes have

$$w_i(b, d) = p_i^{\frac{1+b+d}{1+b}},$$

as in d -average b -redundancy.

One can think of w'_i as representing an invertible function of maximal b -redundancy,

$$w'_i = \left[\sum_{j=1}^n p_j^{\frac{1}{1+b}} \right]^{-1} \cdot 2^{\max_i r_b(i)},$$

where, at any given point of the algorithm, $r_b(i) = l_i - l_i^\dagger$ uses the depth of item i in its interim code tree as the value l_i . Note that only $r_b(i)$ is variable; the denominator term of w'_i is a result of not normalizing the weights at the start of the algorithm. In a similar manner, w''_i represents the probability of maximal b -redundancy $P_X[r_b(X) = \max_j r_b(j)]$.

To implement this algorithm, we let $w'_i = p_i^{\frac{1}{1+b}}$ and $w''_i = p_i$ for the initial case. In comparing items j and k , we consider them as lexicographically ordered pairs — e.g., $w_j = (w'_j, w''_j)$ — so that $w_j \geq w_k$ if and only if either $w'_j > w'_k$ or if $w'_j = w'_k$ and $w''_j \geq w''_k$, as in [5]. In combining items j and k (where $w_j \geq w_k$ as described), the new item will have $\tilde{w}'_j = 2w'_j = 2 \cdot \max(w'_j, w'_k)$. If $w'_j > w'_k$, then $\tilde{w}'_j = w'_j$. Otherwise, $\tilde{w}'_j = w'_j + w'_k$. That is,

$$\tilde{w}_j = \begin{cases} (2w'_j, w''_j) & \text{if } w'_j > w'_k \\ (2w'_j, w''_j + w''_k) & \text{otherwise.} \end{cases}$$

The reasons for this are easily seen if we view w_i as the representation of maximal redundancy and probability this maximal redundancy is achieved. Take the maximum and add 1 for the additional bit of the codeword (multiplying w'_i by 2). Then, if the redundancies are identical, add their probabilities (w''_i). Otherwise, take the probability of the maximal redundancy.

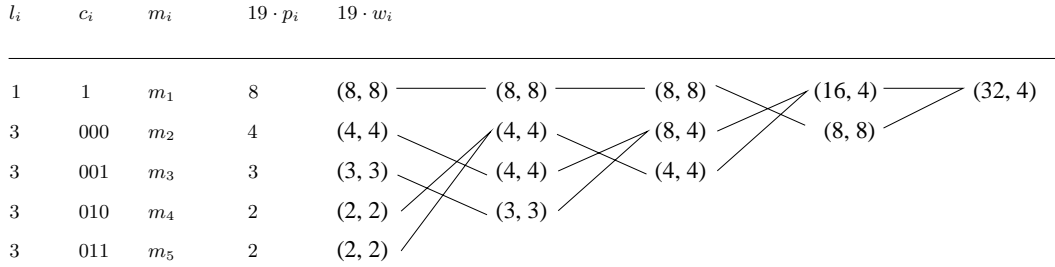


Fig. 3. Algebraic maximal redundancy coding, $\mathbf{p} = \frac{1}{19} \cdot (8, 4, 3, 2, 2)$ (bottom-merge)

This combining method is a Huffman algebra, satisfying the properties introduced in [5]. The Huffman combining criterion is shown by example in Figure 3. The remaining weight pair after coding, $(\frac{32}{19}, \frac{4}{19})$, indicates a maximal redundancy of $\log_2 \frac{32}{19}$ and a probability of $\frac{4}{19}$ that this redundancy is achieved.

We now show that ties in the w pairs imply ties in all terms of the expansion presented in (7), or, equivalently, for d th exponential redundancy for all $d \in [D, +\infty)$ where D is some (unspecified) constant.

Theorem 1: If there is a tie in the above w pairs, there is a tie in all terms of the corresponding d expansion.

Proof: Consider two tied pairs. Note that, in each,

$$w'_i \geq w''_i \frac{1}{1+b} \quad (8)$$

because this holds with equality in leaf nodes and the inequality is preserved in the merge step, since $2 \cdot \max(a, b) \geq a + b \geq \max(a, b)$ for $a, b \geq 0$. If inequality (8) holds without equality for the tied pairs, neither node on the corresponding code tree can be a leaf node, and, due to ordering for the combination step, their four children must be identically weighed. However, this fact can be invoked inductively for either pair of children, also tied, and thus such a tree could not be finite. Therefore, tied pairs arise only in cases for which the inequality holds with equality. Thus, they must be leaf nodes or nodes with two identically-weighted children. Inductively, this means the subtrees must be composed of leaf nodes that are *relatively dyadic*, that is, are dyadic when multiplied by a nontrivial common constant. Thus they are equal in all terms, which is what we set out to show. ■

One can use bottom-merge or top-merge coding so that the algorithm is deterministic. If one uses top-merge coding — that is, favoring combined items over single items with identical weight [13] — one actually need not keep track of the second term; the top-merge algorithm behaves identically without considering this term. This variant, illustrated in Figure 4, is actually a special case of the tree-height measure problem mentioned above. However, if we wish to assure that the solution has minimum variance, the algebraic method is needed.

V. BOUNDS

One can easily see that if we relax the integer constraint on length for minimizing d -average b -redundancy, the real-valued solution is not l^\dagger , but some different l^\ddagger . By substituting the solution in (3), we find

$$l_i^\ddagger = -\omega \cdot \log_2 p_i + \log_2 \sum_{j \in \mathcal{X}} p_j^\omega$$

$$\text{where } \omega = \frac{1+b+d}{(1+b)(1+d)} = 1 - \frac{bd}{(1+b)(1+d)}.$$

Note that when the values of b and d are exchanged, the ideal solution remains the same. This problem thus has a high degree of symmetry. However, because the problem itself is not symmetric, the symmetry of integer solutions is not perfect, as we can see in Figure 5.

Using this and the Shannon code [18] analogue $[l_i^\ddagger]$, we can find bounds for the optimal DABR when $b \geq -1$, $d \geq -1$, and $b + d \geq -1$:

$$0 \leq R_{b,d}(\mathbf{p}, \mathbf{l}_{b,d}^*(\mathbf{p})) - \alpha b (H_\omega(p) - H_\alpha(p)) < 1$$

where we recall $\omega = \frac{1+b+d}{(1+b)(1+d)}$ and $\alpha = \frac{1}{1+b}$, the subscript of Rényi entropy in (2). As with exponential Huffman coding, equality holds iff the ideal solution l^\ddagger has all integer lengths. For $b = +\infty$ and $d = 0$, this results in the well-known Shannon bounds. For $b = 0$, it reduces to a normalized version of an inequality in [8]. With a different normalization, this inequality relates to Rényi's gain of information of order α , a generalization of relative entropy [21]. This is not surprising given the relationship between relative entropy and Huffman coding noted by Longo and Galasso [22].

Due to the reduction to exponential Huffman coding, more sophisticated redundancy results may be applied if desired. The bounds given by Blumer and McEliece [23] apply to the exponential case but appear as solutions to related problems rather than in closed form. Taneja [24] gave closed-form bounds using an alternative definition of redundancy.

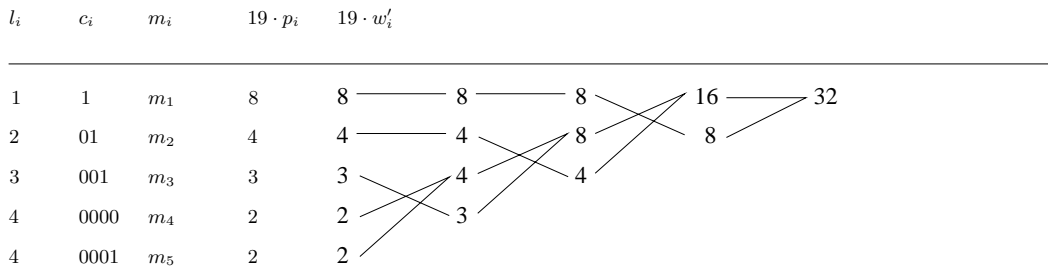


Fig. 4. Top-merge maximal redundancy coding, $\mathbf{p} = \frac{1}{19} \cdot (8, 4, 3, 2, 2)$ (single variable)

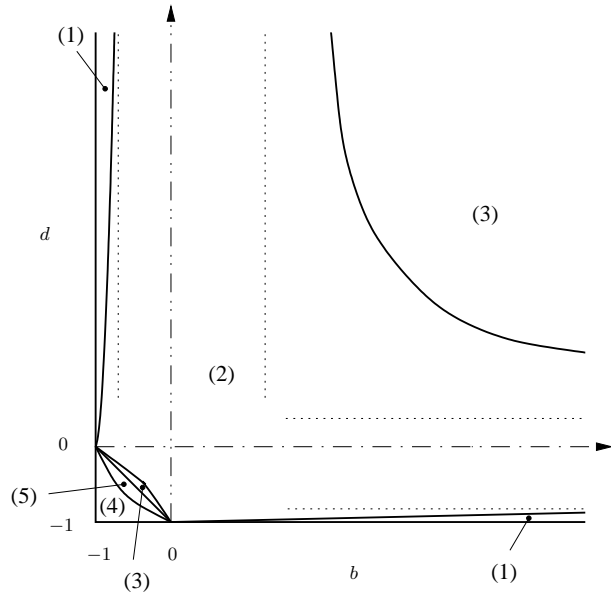


Fig. 5. The parameter space for minimal DABR coding for $\mathbf{p} = (0.58, 0.12, 0.11, 0.1, 0.09)$. Each region represents a set of problems with the same solution. On the transition curves (solid), multiple solutions are optimal. The five distinct solution regions are (1) $\mathbf{l} = (1, 2, 3, 4, 4)$, (2) $\mathbf{l} = (1, 3, 3, 3, 3)$, (3) $\mathbf{l} = (2, 2, 2, 3, 3)$, (4) $\mathbf{l} = (4, 4, 3, 2, 1)$, and (5) $\mathbf{l} = (3, 3, 2, 2, 2)$. The dotted lines within the parameter space indicate the $(+\infty)$ asymptotic behavior of the limits between regions. Note that $b + d + 1 = 0$ divides nonincreasing length vectors from nondecreasing. Also note the high degree of symmetry; the imperfection of this symmetry is best illustrated by the different asymptotes.

VI. CONCLUSION

A two-dimensional framework is demonstrated to encompass examples considered by Parker [4] — classical Huffman coding [1], the exponential variant proposed by Campbell [7], and the d th exponential redundancy problem proposed by Nath [8]. These examples, along with all problems within the framework, are solvable by Huffman-like algorithms. The maximal redundancy problem proposed by Drmota and Szpankowski [9], [19] is shown to be optimized by its equivalence to another example

considered by Parker; the top-merge version of the algorithm in particular additionally optimizes d th exponential redundancy for large d . A better solution — one minimizing codeword length variance among such optimal codes — is suggested by and is developed from the two-dimensional framework introduced here. All algorithms discussed are Huffman-like and thus linear-time given sorted input, unlike the original algorithm proposed for maximal redundancy.

It is unclear whether all nontrivial problems within Parker’s more general framework are covered by this seemingly more specific framework and trivial extensions thereof. Such analysis, building upon Parker’s work, could be a basis for further research. Extending this algorithm to alphabetic codes (alphabetic search trees) could also be explored. For nonnegative exponents ($d \geq 0$), this framework is a trivial extension of [3], but negative exponents might provide more of a challenge.

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